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OBSERVATIONS ON THE INTEGRAL SOLUTIONS OF THE TERNARY QUADRATIC EQUATION

$$x^2 + y^2 = z^2 + 10$$

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Abstract:
This paper illustrates the process of obtaining different sets of non-zero distinct integer solutions to the non-homogeneous ternary quadratic Diophantine equation given by $x^2 + y^2 = z^2 + 10$

Keywords: Non-homogeneous quadratic, Ternary quadratic, Integer solutions.

Introduction:
It is known that Diophantine equations with multi-degree and multiple variables are rich invariety [1,2].While searching for the collection of second-degree equations with three unknowns, the authors came across the papers [3,4,5,6,7] in which the authors obtained integer solutions to the ternary quadratic equation $x^2 + y^2 = z^2 + N, N = 1, \pm 4, 8$.

The above papers motivated us for obtaining non-zero distinct integer solutions to the above equation for other values to N. This communication illustrates the process of obtaining different sets of non-zero distinct integer solutions to the non-homogeneous ternary quadratic Diophantine equation given by $x^2 + y^2 = z^2 + 10$

Method of analysis:
The non-homogeneous ternary quadratic Diophantine equation under consideration is

$$x^2 + y^2 = z^2 + 10 \qquad \ldots (1)$$

The process of obtaining different sets of integer solutions to (1) is illustrated below:

Illustration 1:
The choice

$$z = x + k, k \geq 0 \qquad \ldots (2)$$

in (1) leads to the parabola

$$y^2 = k^2 + 2kx + 10 \qquad \ldots (3)$$

It is possible to choose k, x so that the R.H.S. of (3) is a perfect square and the value of y is obtained. Substituting the values of k, x in (2),the corresponding value of z satisfying (1) is obtained. For simplicity and brevity, a few examples are given in

Table 1: Example

k	x	y	z
1	$2n^2+6n-1$	$2n+3$	$2n^2+6n$
3	$\frac{1}{6}\left(n^4-2n^3+11n^2-10n+6\right)$	n^2-n+5	$\frac{1}{6}\left(n^4-2n^3+11n^2-10n+24\right)$
5	$10n^2+10n-1$	$10n+5$	$10n^2+10n+4$

Illustration 2:

The Substitution of the linear transformation

$$x = y + k, (k \geq 0) \qquad \ldots (4)$$

in (1) leads to the pell equation

$$2y^2 = z^2 - (k^2 + 2ky) + 10 \qquad \ldots (5)$$

which is solvable only for special values k, . For example, considering the value k to be 2 in (5), one obtains the positive pell equation

$$Y^2 = 2z^2 + 16,\ Y = 2y + 2 \qquad \ldots (6)$$

whose smallest positive integer solution is $z_0 = 8,\ Y_0 = 12$

To obtain the other solution of (6) considerthe pell equation

$$Y^2 = 2z^2 + 1 \qquad \ldots (7)$$

whose smallest positive integer solution is $(\tilde{z}_0, \tilde{Y}_0) = (2,3)$

If $(\tilde{x}_n, \tilde{Y}_n)$ represents the general solution of (7), then it is given by

$$\tilde{z}_n = \frac{1}{2\sqrt{2}} g_n,\ \tilde{Y}_n = \frac{1}{2} f_n$$

where

$$f_n = (3+2\sqrt{2})^{n+1} + (3-2\sqrt{2})^{n+1}$$

$$g_n = (3+2\sqrt{2})^{n+1} - (3-2\sqrt{2})^{n+1}$$

Applying the Brahmagupta lemma between (z_0, Y_0) and $(\tilde{z}_n, \tilde{Y}_n)$, we have

$$z_{n+1} = 4 f_n + 3\sqrt{2} g_n,$$

$$Y_{n+1} = 6 f_n + 4\sqrt{2} g_n$$

$y_{n+1}=3f_n+2\sqrt{2}g_n-1 \quad \left(\because y=\frac{Y-2}{2}\right)$

In view of (4)

$x_{n+1}=3f_n+2\sqrt{2}g_n+1$

The above values of $x_{n+1}, y_{n+1}, z_{n+1}$ represents the general solution to (1).

The recurrence relations satisfied by $z_{n+1}, x_{n+1}, y_{n+1}$ are given by

$z_{n+1}-6z_{n+2}+z_{n+3}=0,\ n=-1,0,1,2,\ldots$ ...(8)

$x_{n+1}-6x_{n+2}+x_{n+3}=-4,\ n=-1,0,1,2,\ldots$ ... (9)

$y_{n+1}-6y_{n+2}+y_{n+3}=4,\ n=-1,0,1,2,\ldots$ ... (10)

Some numerical examples satisfying (1) for $k=2$ are given in Table 2 below:

Table 2: Numerical examples

n	z_{n+1}	y_{n+1}	z_{n+1}
-1	8	5	7
0	48	33	35
1	280	197	199
2	1632	1153	1155
3	9512	6725	6727
4	55440	39201	39203
5	323128	228485	228487

Observations:

1. All the values of z_{n+1} are even, where as the values of x_{n+1}, y_{n+1} are odd.
2. $z_{n+1}\equiv 0 \pmod{8}$
3. $x_{3n-3}, x_{3n-2}\equiv 0 \pmod{7}$
4. **A few interesting relations among the solutions:**
   - ❖ $z_{n+2}-3z_{n+1}-4x_{n+1}+4=0$
   - ❖ $z_{n+3}-17z_{n+1}-24x_{n+1}+24=0$
5. **Expressions representing Nasty Numbers:**
   - ❖ $\frac{1}{4}\left(18z_{2n+3}-102z_{2n+2}+48\right)$
   - ❖ $\frac{1}{8}\left(6z_{2n+4}-198z_{2n+2}+96\right)$
6. **Expressions representing Cubical Integers:**
   - ❖ $\frac{1}{4}\left[3z_{3n+4}-17z_{3n+3}+9z_{n+2}-51z_{n+1}\right]$
   - ❖ $\frac{1}{8}\left[z_{3n+5}-33z_{3n+3}+3z_{n+3}-99z_{n+1}\right]$

7. **Expressions representing Bi-Quadratic Integers:**

- $\frac{1}{4}\left[3z_{4n+5}-17z_{4n+4}+12z_{2n+3}-68z_{2n+2}+24\right]$
- $\frac{1}{8}\left[z_{4n+6}-33z_{4n+4}+4z_{2n+4}-132z_{2n+2}+48\right]$

8. **Expressions representing Quintic Integers:**

- $\frac{1}{4}\left[3z_{5n+6}-17z_{5n+5}+15z_{3n+4}-85z_{3n+3}+30z_{n+2}-170z_{n+1}\right]$
- $\frac{1}{8}\left[z_{5n+7}-33z_{5n+5}+5z_{3n+5}-165z_{3n+3}+10z_{n+3}-330z_{n+1}\right]$

9. **Employing linear combinations among the solutions, one obtains integer solutions to different choices of Hyperbolas:**

Choice: 1

Let $Y_n=3z_{n+2}-17z_{n+1},\ X_n=6z_{n+1}-z_{n+2}\Rightarrow$

$Y_n{}^2-8X_n{}^2=64$, a hyperbola.

Choice: 2

Let $Y_n=z_{n+3}-33z_{n+1}, X_n=35z_{n+1}-z_{n+3}\Rightarrow$

$9Y_n{}^2-8X_n{}^2=2304$, a hyperbola.

10. **Employing linear combinations among the solutions, one obtains integer solutions to different choices of Parabolas:**

Choice: 1

Let $Y_n=3z_{2n+3}-17z_{2n+2},\ X_n=6z_{n+1}-z_{n+2}\Rightarrow$

$Y_n-2X_n{}^2=8$, a parabola.

Choice: 2

Let $Y_n=z_{2n+4}-33z_{2n+2}, X_n=35z_{n+1}-z_{n+3}\Rightarrow$

$9Y_n-X_n{}^2=144$, a parabola.

Illustration 3:

The Substitution of the linear transformation

$z=k\,x,\ (k\succ 1)$ ... (11)

in (1) leads to the positive pell equation

$y^2=(k^2-1)x^2+10$ ...(12)

which is solvable only for special values k. For example,considering the value k to be 4in (12), one obtains the positive pell equation

$y^2=15x^2+10$ ... (13)

whose smallest positive integer solution is $x_0 = 1, y_0 = 5$

To obtain the other solutions to (13) consider the pell equation

$$y^2 = 15x^2 + 1 \qquad \ldots (14)$$

whose smallest positive integer solution is $(\tilde{x}_0, \tilde{y}_0) = (1,4)$

If $(\tilde{x}_n, \tilde{y}_n)$ represents the general solution of (14), then it is given by

$$\tilde{x}_n = \frac{1}{2\sqrt{15}} g_n,\ \tilde{y}_n = \frac{1}{2} f_n$$

where

$$f_n = \left(4+\sqrt{15}\right)^{n+1} + \left(4-\sqrt{15}\right)^{n+1}$$

$$g_n = \left(4+\sqrt{15}\right)^{n+1} - \left(4-\sqrt{15}\right)^{n+1}$$

Applying the Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, we have

$$x_{n+1} = \frac{1}{2} f_n + \frac{\sqrt{15}}{6} g_n,$$

$$y_{n+1} = \frac{5}{2} f_n + \frac{\sqrt{15}}{2} g_n$$

In view of (11),

$$z_{n+1} = \frac{2}{3}(3 f_n + \sqrt{15} g_n)$$

The above values of $x_{n+1}, y_{n+1}, z_{n+1}$ represents the general solution to (1).

The recurrence relations satisfied by $x_{n+1}, y_{n+1}, z_{n+1}$ are given by

$$x_{n+1} - 8x_{n+2} + x_{n+3} = 0,\ n = -1,0,1,2,\ldots \qquad \ldots (15)$$

$$y_{n+1} - 8y_{n+2} + y_{n+3} = 0,\ n = -1,0,1,2,\ldots \qquad \ldots (16)$$

$$z_{n+1} - 8z_{n+2} + z_{n+3} = 0,\ n = -1,0,1,2,\ldots \qquad \ldots (17)$$

Some numerical examples satisfying(1) for $k = 4$ are given in the Table 3below:

Table 3: Numerical examples

n	x_{n+1}	y_{n+1}	z_{n+1}
-1	1	5	4
0	9	35	36
1	71	275	284
2	559	2165	2236
3	4401	17045	17604
4	34649	134195	138596
5	272791	1056515	1091164

Observations:

1. All the values of x_{n+1}, y_{n+1} are odd, where as the values of z_{n+1} are even.
2. $y_{n+1} \equiv 0 \pmod 5$
3. $x_{n+1} + x_{n+2} \equiv 0 \pmod{10}$
4. **A few interesting relationsamong the solutions:**
   - ❖ $y_{n+1} - x_{n+2} + 4x_{n+1} = 0$
   - ❖ $8y_{n+1} - x_{n+3} + 31x_{n+1} = 0$
5. **Expressions representing Nasty Numbers:**
   - ❖ $(6x_{2n+3} - 42x_{2n+2} + 12)$
   - ❖ $\frac{1}{8}(6x_{2n+4} - 330x_{2n+2} + 96)$
6. **Expressions representing Cubical Integers:**
   - ❖ $[x_{3n+4} - 7x_{3n+3} + 3x_{n+2} - 21x_{n+1}]$
   - ❖ $\frac{1}{8}[x_{3n+5} - 55x_{3n+3} + 3x_{n+3} - 165x_{n+1}]$
7. **Expressions representing Bi-Quadratic Integers:**
   - ❖ $[x_{4n+5} - 7x_{4n+4} + 4x_{2n+3} - 28x_{2n+2} + 6]$
   - ❖ $\frac{1}{8}[x_{4n+6} - 55x_{4n+4} + 4x_{2n+4} - 220x_{2n+2} + 48]$.
8. **Expressions representing Quintic Integers:**
   - ❖ $[x_{5n+6} - 7x_{5n+5} + 5x_{3n+4} - 35x_{3n+3} + 10x_{n+2} - 70x_{n+1}]$
   - ❖ $\frac{1}{8}[x_{5n+7} - 55x_{5n+5} + 5x_{3n+5} - 275x_{3n+3} + 10x_{n+3} - 550x_{n+1}]$
9. **Employing linear combinations among the solutions, one obtains integer solutions to different choices of Hyperbolas:**

   Choice 1:

Let $Y_n = x_{n+2} - 7x_{n+1}, \; X_n = 9x_{n+1} - x_{n+2} \Rightarrow$

$5Y_n^2 - 3X_n^2 = 20$, a hyperbola.

Choice 2:

Let $Y_n = x_{n+3} - 55x_{n+1}, \; X_n = 213x_{n+1} - 3x_{n+3} \Rightarrow$

$15Y_n^2 - X_n^2 = 3840$, a hyperbola.

10. **Employing linear combinations among the solutions, one obtains integer solutions to different choices of parabolas:**

    Choice 1:

Let $Y_n = x_{2n+3} - 7x_{2n+2}, \; X_n = 9x_{n+1} - x_{n+2} \Rightarrow$

$5Y_n - 3X_n^2 = 10$, a parabola.

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Choice 2:

Let $Y_n = x_{2n+4} - 55x_{2n+2}$, $X_n = 213x_{n+1} - 3x_{n+3} \Rightarrow$

$120Y_n - X_n^{\ 2} = 1920$, a parabola.

Illustration 4:

The substitution of the linear transformations

$$z = (k+1)\alpha \ , x = k\alpha \qquad \ldots (18)$$

in (1) leads to the positive pell equation

$$y^2 = (2k+1)\alpha^2 + 10 \qquad \ldots (19)$$

for which the integer solutions exist when k takes particular values.
For example,considering the value of k to be 19 in (19),it gives the positive pell equation

$$y^2 = 39\alpha^2 + 10 \qquad \ldots (20)$$

whose smallest positive integer solution is $\alpha_0 = 1,\ y_0 = 7$

To obtain the other solutions (20) consider the pell equation

$$y^2 = 39\alpha^2 + 1 \qquad \ldots (21)$$

whose smallest positive integer solution is $(\tilde{\alpha}_0, \tilde{y}_0) = (4,25)$

If $(\tilde{\alpha}_n, \tilde{y}_n)$ represents the general solution of (21),then it is given by

$$\tilde{\alpha}_n = \frac{1}{2\sqrt{39}} g_n, \tilde{y}_n = \frac{1}{2} f_n$$

where

$$f_n = \left(25 + 4\sqrt{39}\right)^{n+1} + \left(25 - 4\sqrt{39}\right)^{n+1}$$

$$g_n = \left(25 + 4\sqrt{39}\right)^{n+1} - \left(25 - 4\sqrt{39}\right)^{n+1}$$

Applying the Brahmagupta lemma between (α_0, y_0) and $(\tilde{\alpha}_n, \tilde{y}_n)$,we have

$$\alpha_{n+1} = \frac{1}{2} f_n + \frac{7}{2\sqrt{39}} g_n,$$

$$y_{n+1} = \frac{7}{2} f_n + \frac{\sqrt{39}}{2} g_n$$

In view of (18),

$$z_{n+1} = 10 f_n + \frac{70\sqrt{39}}{39} g_n,$$

$$x_{n+1} = \frac{19}{2} f_n + \frac{133}{2\sqrt{39}} g_n$$

The above values of $x_{n+1}, y_{n+1}, z_{n+1}$ represents the general solutions to (1).

The recurrence relations satisfied by $y_{n+1}, z_{n+1}, x_{n+1}$ are given by

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$y_{n+1} - 50y_{n+2} + y_{n+3} = 0,\ n = -1,0,1,2,\ldots$ ... (22)

$z_{n+1} - 50z_{n+2} + z_{n+3} = 0,\ n = -1,0,1,2,\ldots$ ...(23)

$x_{n+1} - 50x_{n+2} + x_{n+3} = 0,\ n = -1,0,1,2,\ldots$ ... (24)

Some numerical examples satisfying(1) for $k = 19$ are given in Table 4below:

Table 4: Numerical examples

n	y_{n+1}	x_{n+1}	z_{n+1}	α_{n+1}
-1	7	19	20	1
0	331	1007	1060	53
1	16543	50331	52980	2649
2	826819	2515543	2647940	132397
3	41324407	125726819	132344020	6617201
4	2065393531	6283825407	6614553060	330727653
5	103228352143	314065543531	330595308980	16529765449

Observations:

1. All the values x_{n+1}, y_{n+1} are odd, where as the values of z_{n+1} are even.
2. $x_{n+1} \equiv 0 \pmod{19}$
3. $z_{n+1} \equiv 0 \pmod{20}$
4. **A few interesting relations among the solutions:**
   - $4y_{n+1} - \alpha_{n+2} + 25\alpha_{n+1} = 0$
   - $200y_{n+1} - \alpha_{n+3} + 1249\alpha_{n+1} = 0$
5. **Expressions representing Nasty Numbers:**
   - $\frac{1}{20}\left(42\alpha_{2n+3} - 1986\alpha_{2n+2} + 240\right)$
   - $\frac{1}{1000}\left(42\alpha_{2n+4} - 99258\alpha_{2n+2} + 12000\right)$
6. **Expressions representing Cubical Integers:**
   - $\frac{1}{20}\left[7\alpha_{3n+4} - 331\alpha_{3n+3} + 21\alpha_{n+2} - 993\alpha_{n+1}\right]$
   - $\frac{1}{1000}\left[7\alpha_{3n+5} - 16543\alpha_{3n+3} + 21\alpha_{n+3} - 49629\alpha_{n+1}\right]$
7. **Expressions representing Bi-Quadratic Integers:**
   - $\frac{1}{20}\left[7\alpha_{4n+5} - 331\alpha_{4n+4} + 28\alpha_{2n+3} - 1324\alpha_{2n+2} + 120\right]$
   - $\frac{1}{1000}\left[7\alpha_{4n+6} - 16543\alpha_{4n+4} + 28\alpha_{2n+4} - 66172\alpha_{2n+2} + 6000\right]$

8. Expressions representing Quintic Integers:

- $\frac{1}{20}\left[7\alpha_{5n+6}-331\alpha_{5n+5}+35\alpha_{3n+4}-1655\alpha_{3n+3}+70\alpha_{n+2}-3310\alpha_{n+1}\right]$
- $\frac{1}{1000}\left[7\alpha_{5n+7}-16543\alpha_{5n+5}+35\alpha_{3n+5}-82715\alpha_{3n+3}+70\alpha_{n+3}-165430\alpha_{n+1}\right]$

9. Employing linear combinations among the solutions, one obtains integer solutions to different choices of Hyperbolas:

Choice 1:

Let $Y_n=7\alpha_{n+2}-331\alpha_{n+1},\ X_n=53\alpha_{n+1}-\alpha_{n+2}\Rightarrow$

$Y_n^{\,2}-39X_n^{\,2}=1600$, a hyperbola.

Choice 2:

Let $Y_n=7\alpha_{n+3}-16543\alpha_{n+1},\ X_n=2649\alpha_{n+1}-\alpha_{n+3}\Rightarrow$

$Y_n^{\,2}-39X_n^{\,2}=4000000$, a hyperbola.

10. Employing linear combinations among the solutions, one obtains integer solutions to different choices of Parabolas:

Choice 1:

Let $Y_n=7\alpha_{2n+3}-331\alpha_{2n+2},\ X_n=53\alpha_{n+1}-\alpha_{n+2}\Rightarrow$

$20Y_n-39X_n^{\,2}=800$, a parabola.

Choice 2:

Let $Y_n=7\alpha_{2n+4}-16543\alpha_{2n+2},\ X_n=2649\alpha_{n+1}-\alpha_{n+3}\Rightarrow$

$1000Y_n-39X_n^{\,2}=2000000$, a parabola.

Illustration 5:

The Substitution of the linear transformations

$x=u+h,\ y=u-h,\ u\neq h\neq 0$ ...(25)

in (1) leads to the pell equation

$z^2=2u^2+(2h^2-10)$ ...(26)

which is solvable only for special values h.. For example, considering the value h to be 1 in (26), one obtains the negative pell equation

$z^2=2u^2-8$ ...(27)

whose smallest negative integer solution is $u_0=2,\ z_0=0$

To obtain the other solutions to (27) consider the pell equation

$z^2=2u^2+1$ ...(28)

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whose smallest positive integer solution is $(\tilde{u}_0, \tilde{z}_0) = (2,3)$

If $(\tilde{u}_n, \tilde{z}_n)$ represents the general solutions of (28), then it is given by

$$\tilde{u}_n = \frac{1}{2\sqrt{2}} g_{n,} \; \tilde{z}_n = \frac{1}{2} f_n$$

where

$$f_n = (3+2\sqrt{2})^{n+1} + (3-2\sqrt{2})^{n+1}$$

$$g_n = (3+2\sqrt{2})^{n+1} - (3-2\sqrt{2})^{n+1}$$

Applying the Brahmagupta lemma between (u_0, z_0) and $(\tilde{u}_n, \tilde{z}_n)$, we have

$$u_{n+1} = f_n,$$

$$z_{n+1} = \sqrt{2} g_n$$

In view of (25),

$$x_{n+1} = f_n + 1,$$

$$y_{n+1} = f_n - 1$$

The above values of $x_{n+1}, y_{n+1}, z_{n+1}$ represents the general solutions to (1).

The recurrence relations satisfied by $z_{n+1}, x_{n+1}, y_{n+1}$ are given by

$$z_{n+1} - 6z_{n+2} + z_{n+3} = 0, \; n = -1,0,1,2,\ldots \qquad \ldots(29)$$

$$x_{n+1} - 6x_{n+2} + x_{n+3} = -4, \; n = -1,0,1,2,\ldots \qquad \ldots(30)$$

$$y_{n+1} - 6y_{n+2} + y_{n+3} = 4, \; n = -1,0,1,2,\ldots \qquad \ldots(31)$$

Some numerical examples satisfying (1) for $h = 1$ are given in Table 5 below:

Table 5: Numerical examples

n	z_{n+1}	x_{n+1}	y_{n+1}	u_{n+1}
-1	0	3	1	2
0	8	7	5	6
1	48	35	33	34
2	280	199	197	198
3	1632	1155	1153	1154
4	9512	6727	6725	6726
5	55440	39203	39201	39202

Observations:

1. All the values z_{n+1} areeven, where as the values of x_{n+1}, y_{n+1} are odd.
2. $z_{n+1} \equiv 0 \pmod{8}$ when $n = 0,1,2,3,\ldots$
3. **A few interesting relations among the solutions:**
    - ❖ $2z_{n+1} - u_{n+2} + 3u_{n+1} = 0$
    - ❖ $12z_{n+1} - u_{n+3} + 17u_{n+1} = 0$

4. **Expressions representing Nasty Numbers:**
   - $\left(6u_{2n+2}+12\right)$
   - $\left(36u_{2n+3}-6u_{2n+4}+12\right)$
5. **Expressions representing Cubical Integers:**
   - $\left[u_{3n+3}-3u_{n+1}\right]$
   - $\left[6u_{3n+4}-u_{3n+5}+18u_{n+2}-3u_{n+3}\right]$
6. **Expressions representing Bi-Quadratic Integers:**
   - $\left[u_{4n+4}+4u_{2n+2}+6\right]$
   - $\left[6u_{4n+5}-u_{4n+6}+24u_{2n+3}-4u_{2n+4}+6\right]$
7. **Expressions representing Quintic Integers:**
   - $\left[u_{5n+5}+5u_{3n+3}+10u_{n+1}\right]$
   - $\left[6u_{5n+6}-u_{5n+7}+30u_{3n+4}-5u_{3n+5}+60u_{n+2}-10u_{n+3}\right]$
8. **Employing linear combinations solutions, one obtains integer solutions to different choices of Hyperbolas:**

Choice 1:

Let $Y_n=u_{n+1},\ X_n=u_{n+2}-3u_{n+1}\Rightarrow$

$8Y_n^{\,2}-X_n^{\,2}=32$, a hyperbola.

Choice 2:

Let $Y_n=u_{n+1},\ X_n=u_{n+3}-17u_{n+1}\Rightarrow$

$288Y_n^{\,2}-X_n^{\,2}=1152$, hyperbola.

9. **Employing linear combinations solutions, one obtains integer solutionsto different choices of Parabolas:**

Choice 1:

Let $Y_n=u_{2n+2},\ X_n=u_{n+2}-3u_{n+1}\Rightarrow$

$8Y_n-X_n^{\,2}=16$, a parabola.

Choice 2:

Let $Y_n=u_{2n+2},\ X_n=u_{n+3}-17u_{n+1}\Rightarrow$

$288Y_n-X_n^{\,2}=576$, a parabola.

Conclusion:

In this paper, we have presented different patterns of non-homogeneous ternary quadratic diophantine equation. In conclusion, one may search for non-zero distinct integer solutions to other choices of homogeneous or non-homogeneous ternary quadratic diophantine equations along with their corresponding properties.

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